Pointwise Values and Fundamental Theorem in the Algebra of Asymptotic Functions

Todor. D. Todorov

Introduction: The algebra of asymptotic functions ${}^\rho\mathcal{E}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^d$ was introduced by M. Oberguggenberger [8] and the author of this paper [14] in the framework of A. Robinson's nonstandard analysis. It can be described as a differential associative and commutative ring (of generalized functions) which is an algebra over the field of A. Robinson's asymptotic numbers ${}^\rho\mathbb{C}$ (A. Robinson [13] and A. H. Lightstone and A. Robinson [6]). Moreover, ${}^\rho\mathcal{E}((\Omega)$ is supplied with the chain of imbeddings:

$$\mathcal{E}(\Omega) \subset \mathcal{D}'(\Omega) \subset {}^\rho\mathcal{E}(\Omega)\,, \tag{0.1}$$

where $\mathcal{E}(\Omega)$ denotes the differential ring of the C^∞-functions (complex valued) on $\Omega, \mathcal{D}(\Omega)$ denotes the differential ring of the functions in $\mathcal{E}(\Omega)$ with compact support in Ω and $\mathcal{D}'(\Omega)$ denotes the differential linear space of Schwartz distributions on Ω. Here $\mathcal{E}(\Omega) \subset \mathcal{D}'(\Omega)$ is the usual imbedding in the sense of distribution theory (V. Vladimirov [16]). The imbedding $\mathcal{D}'(\Omega) \subset {}^\rho\mathcal{E}(\Omega)$, constructed in (M. Oberguggenberger and T. Todorov [10]), preserves all linear operations, including partial differentiation of any order, and the pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. Finally, the imbedding $\mathcal{E}(\Omega) \subset {}^\rho\mathcal{E}(\Omega)$ preserves all differential ring operations. In addition, if T^d denotes the usual topology on $\mathbb{R}^d$, then the family $\{{}^\rho\mathcal{E}(\Omega)\}_{\Omega\in T^d}$ is a sheaf of differential rings and the imbeddings in (0.1) are sheaf preserving. That means, in particular, that the restriction $F|\Omega'$ is a well defined element in ${}^\rho\mathcal{E}(\Omega')$ for any $F \in {}^\rho\mathcal{E}(\Omega)$ and any open $\Omega' \subseteq \Omega$. On these grounds we consider the asymptotic functions in ${}^\rho\mathcal{E}(\Omega)$ as "generalized functions on Ω": they are "generalized" functions (rather than "classical") because they are not mappings from Ω into $\mathbb{C}$. On the other hand, they are still "functions" (although, "generalized" ones) : **a)** because of the imbeddings (0.1) and **b)** because of the sheaf properties mentioned above. As a result, ${}^\rho\mathcal{E}(\Omega)$, supplied with the imbeddings (0.1), offers a solution to

the problem for multiplication of Schwartz distributions - similar to the solution to the same problem presented not long ago by J. F. Colombeau (and other authors) in the framework of standard analysis (J. F. Colombeau [1]-[2]). Our interest in the algebra ${}^{\rho}\mathcal{E}(\Omega)$ is due to its importance for the theory of partial differential equations - linear (T. Todorov [15]) and nonlinear (M. Oberguggenberger [9]).

The purpose of this paper is to show that ${}^{\rho}\mathcal{E}(\Omega)$ is isomorphic to a subring of the ring $({}^{\rho}\mathbb{C})^{{}^{\rho}\tilde{\Omega}}$ of the pointwise functions from ${}^{\rho}\tilde{\Omega}$ into ${}^{\rho}\mathbb{C}$. Here ${}^{\rho}\mathbb{R}$ and ${}^{\rho}\mathbb{C}$ are the fields of the A. Robinson's asymptotic numbers (A. Robinson [13]), real and complex, respectively, ${}^{\rho}\mathbb{R}^d$ is the corresponding vector space, and ${}^{\rho}\tilde{\Omega} = \Omega + \mathcal{I}({}^{\rho}\mathbb{R}^d)$ denotes the set of the points in ${}^{\rho}\mathbb{R}^d$ of the form $x + h$, where $x \in \Omega$ and h is an infinitesimal point in ${}^{\rho}\mathbb{R}^d$. As a result, every asymptotic function in ${}^{\rho}\mathcal{E}(\Omega)$ can be identified with a unique pointwise function defined on ${}^{\rho}\tilde{\Omega}$ taking values in ${}^{\rho}\mathbb{C}$. In the particular case of classical smooth functions our evaluation coincides with the usual evaluation. Moreover, since ${}^{\rho}\mathcal{E}(\Omega)$ contains a copy of the space of Schwartz distributions, it follows that the distributions are also pointwise functions taking values in ${}^{\rho}\mathbb{C}$. We also prove the Fundamental Theorem of Integration Calculus in ${}^{\rho}\mathcal{E}(\Omega)$ and show that the set of the scalars of ${}^{\rho}\mathcal{E}(\Omega)$ coincides exactly with the field ${}^{\rho}\mathbb{C}$ and the set of the real scalars of ${}^{\rho}\mathcal{E}(\Omega)$ coincides exactly with the field ${}^{\rho}\mathbb{R}$.

Recall that if L is a differential linear space, then the set of the scalars $\mathcal{S}$ in L consists of the elements F in L with zero partial first derivatives $\partial^{\alpha} F = 0, |\alpha| = 1$, in symbols, $\mathcal{S} = \{f \in L : \nabla F = 0\}$. If, in addition, S is supplied with complex conjugation, then the set of the real scalars in L is given by $\mathrm{Re}(\mathcal{S}) = \{\pm |F| : F \in \mathcal{S}\}$.

The theory of asymptotic functions is similar to, but somewhat different from, J. F. Colombeau's nonlinear theory of generalized functions, where we also have a chain of imbeddings $\mathcal{E}(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega)$ with similar properties. Here $\mathcal{G}(\Omega)$ denotes a typical algebra of J. F. Colombeau's generalized functions (J. F. Colombeau [1]-[2]). The set of the scalars $\bar{\mathbb{C}}$, and the real scalars $\bar{\mathbb{R}}$ of $\mathcal{G}(\Omega)$ are called J. F. Colombeau's generalized numbers (J. F. Colombeau [2], 2.1). Recently, M.Oberguggenberger and M. Kunzinger [11] proved that the algebras of the type $\mathcal{G}(\Omega)$ can be canonically imbedded in the ring $\bar{\mathbb{C}}^{\tilde{\Omega}_c}$ of pointwise functions from $\tilde{\Omega}_c$ into $\bar{\mathbb{C}}$, where $\tilde{\Omega}_c$ is a particular set of generalized points such that $\Omega \subset \tilde{\Omega}_c \subset \bar{\mathbb{R}}^d$. Thus, the generalized functions in $\mathcal{G}(\Omega)$ can also be viewed as pointwise functions.

The difference between the algebra of asymptotic functions ${}^{\rho}\mathcal{E}(\Omega)$ and J. F. Colombeau's

algebras of generalized functions of the type $\mathcal{G}(\Omega)$ can be best expressed by comparing their scalars:

a) The set of the scalars $\bar{\mathbb{C}}$ of $\mathcal{G}(\Omega)$ is a non-archimedean ring with zero divisors, containing a copy of $\mathbb{C}$. In contrast, the set of the scalars ${}^\rho\mathbb{C}$ of ${}^\rho\mathcal{E}(\Omega)$ is an algebraically closed non-archimedean field, containing a copy of $\mathbb{C}$. In addition, ${}^\rho\mathbb{C}$ is Cantor complete in the sense that every sequence of closed circles in ${}^\rho\mathbb{C}$ with the finite intersection property has a non-empty intersection in ${}^\rho\mathbb{C}$. Finally, ${}^\rho\mathbb{C}$ contains a canonical copy of the T. Levi-Cività field of asymptotic series with complex coefficients (that property gives the name "asymptotic numbers" for the elements of ${}^\rho\mathbb{C}$).

b) The set of the real scalars $\bar{\mathbb{R}}$ of $\mathcal{G}(\Omega)$ is a partially ordered non-archimedean ring with zero divisors containing a copy of $\mathbb{R}$. In contrast, the set of the real scalars ${}^\rho\mathbb{R}$ of ${}^\rho\mathcal{E}(\Omega)$ is a real closed non-archimedean field extension of $\mathbb{R}$ which is Cantor complete in the sense that every sequence of closed intervals in ${}^\rho\mathbb{R}$ with the finite intersection property has a non-empty intersection in ${}^\rho\mathbb{R}$. Moreover, ${}^\rho\mathbb{R}$ contains a canonical copy of the T. Levi-Cività field of asymptotic series with real coefficients (A. H. Lightstone and A. Robinson [6], p.93).

The improvement of the properties of the scalars ${}^\rho\mathbb{R}$ and ${}^\rho\mathbb{R}$ of ${}^\rho\mathcal{E}(\Omega)$ compared with the properties of the scalars $\bar{\mathbb{C}}$ and $\bar{\mathbb{R}}$ in J. F. Colombeau's theory is the main reason to involve the methods of nonstandard analysis in our approach.

1 Preliminaries: Asymptotic Numbers, Asymptotic Vectors and Asymptotic Functions

Our framework is a nonstandard model of the real numbers $\mathbb{R}$, with degree of saturation larger than card($\mathbb{N}$). Throughout this paper Ω denotes an open subset of $\mathbb{R}^d$. We denote by ${}^*\mathbb{R}$, ${}^*\mathbb{R}_+$, ${}^*\mathbb{C}$, ${}^*\mathcal{E}(\Omega)$ and ${}^*\mathcal{D}(\Omega)$ the nonstandard extensions of $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$, $\mathcal{E}(\Omega)$ and $\mathcal{D}(\Omega)$, respectively. In particular, ${}^*\mathcal{E}(\Omega)$ consists of pointwise functions of the form f: ${}^*\Omega \to {}^*\mathbb{C}$. If X is a set of complex numbers or a set of (standard) functions, then ${}^*X$ will be its nonstandard extension and if $f : X \to Y$ is a (standard) mapping, then ${}^*f : {}^*X \to {}^*Y$ will be its nonstandard extension. For integration in ${}^*\mathbb{R}^d$ we use the $*$-Lebesgue integral. For a very short introduction

to nonstandard analysis we refer to the Appendix in T. Todorov [15]. For a more detailed exposition we recommend T. Lindstrøm [5], where the reader will find many references to the subject.

The purpose of this section is to present a summary of those definitions and results of *Nonstandard Asymptotic Analysis* which we need for our exposition. For more details we refer to A. Robinson [13], A.H. Lightstone and A. Robinson [6], W.A.J. Luxemburg [7], Li Bang-He [4] and V. Pestov [12]. For the algebra of asymptotic functions and its applications we refer to M. Oberguggenberger [8], T. Todorov [14], R. F. Hoskins and J. Sousa Pinto [3], M. Oberguggenberger and T. Todorov [10], M. Oberguggenberger [9] and T. Todorov [15].

Part 1. *Asymptotic Numbers:* Let $\rho \in {}^*\mathbb{R}$ be a positive infinitesimal. We shall keep ρ fixed in what follows. Following A. Robinson [13] we define the field of A. Robinson's complex *ρ-asymptotic numbers* as the factor space ${}^\rho\mathbb{C} = \mathcal{M}_\rho({}^*\mathbb{C})/\mathcal{N}_\rho({}^*\mathbb{C})$, where

$$\mathcal{M}_\rho({}^*\mathbb{C}) = \{\xi \in {}^*\mathbb{C} : |\xi| < \rho^{-n} \quad \text{for some} \quad n \in \mathbb{N}\}, \tag{1.1}$$

$$\mathcal{N}_\rho({}^*\mathbb{C}) = \{\xi \in {}^*\mathbb{C} : |\xi| < \rho^{n} \quad \text{for all} \quad n \in \mathbb{N}\}, \tag{1.2}$$

are the sets of the r-moderate and ρ-null numbers in ${}^*\mathbb{C}$, respectively. We define the embedding $\mathbb{C} \subset {}^\rho\mathbb{C}$ by $c \to q(c)$, where $q : \mathcal{M}_\rho({}^*\mathbb{C}) \to {}^\rho\mathbb{C}$ is the quotient mapping. If $\mathcal{A} \subseteq {}^*\mathbb{C}$, we let

$$\mathcal{M}_\rho(\mathcal{A}) = \{\xi \in \mathcal{A} : |\xi| < \rho^{-n} \quad \text{for some} \quad n \in \mathbb{N}\}, \tag{1.3}$$

and ${}^\rho\mathcal{A} = q[\mathcal{M}_\rho(\mathcal{A})]$. The set ${}^\rho\mathcal{A} \subseteq {}^\rho\mathbb{R}$ is called the ρ-extension of $\mathcal{A}$ in ${}^\rho\mathbb{R}$. In the particular case $\mathcal{A} = {}^*A$, where $A \subseteq \mathbb{C}$, we shall write simply ${}^\rho A$ instead of the more precise ${}^\rho({}^*A)$. The field of *A. Robinson's real ρ-asymptotic numbers* is defined by ${}^\rho\mathbb{R} = q[\mathcal{M}_\rho({}^*\mathbb{R})]$. The imbedding $\mathbb{R} \subset {}^\rho\mathbb{R}$ is defined by $r \to q(r)$. The asymptotic number $s = q(\rho)$ is the scale of ${}^\rho\mathbb{R}$; it is a positive infinitesimal, i.e. $s \in {}^\rho\mathbb{R}$, $s > 0$ and $s \approx 0$. Finally, we observe that if $\alpha \in {}^*\mathbb{C}$, then $\alpha \in \mathcal{M}_\rho({}^*\mathbb{C})$ iff $|\alpha| \in \mathcal{M}_\rho({}^*\mathbb{R})$ and, similarly, $\alpha \in \mathcal{N}_\rho({}^*\mathbb{C})$ iff $|\alpha| \in \mathcal{N}_\rho({}^*\mathbb{R})$.

It is easy to check that $\mathcal{N}_\rho({}^*\mathbb{C})$ is a maximal ideal in $\mathcal{M}_\rho({}^*\mathbb{C})$ and hence ${}^\rho\mathbb{C}$ is a field. Also ${}^\rho\mathbb{R}$ is a real closed totally ordered nonarchimedean field (since ${}^*\mathbb{R}$ is a real closed totally ordered field) containing $\mathbb{R}$ as a totally ordered subfield. Thus,

it follows that ${}^\rho\mathbb{C} = {}^\rho\mathbb{R}(i)$ is an algebraically closed field, where $i = \sqrt{-1}$. Finally ${}^\rho\mathbb{R}$ is Cantor complete in the sense that every sequence of closed intervals in ${}^\rho\mathbb{R}$ with the finite intersection property has a non-empty intersection (since ${}^*\mathbb{R}$ has this property). Unless it is said otherwise, we shall always topologize ${}^\rho\mathbb{R}$ and ${}^\rho\mathbb{C}$ by the order topology in ${}^\rho\mathbb{R}$. The sequence of intervals:

$$I_n = \{x \in {}^\rho\mathbb{R} : -s^n < x < s^n\}, \tag{1.4}$$

forms a base for the neighborhoods of the zero in the *interval topology* in ${}^\rho\mathbb{R}$.

Part 2. *Asymptotic Vectors*: Let ${}^*\mathbb{R}^d$ and ${}^\rho\mathbb{R}^d$ be the corresponding d-dimensional vector spaces over the fields ${}^*\mathbb{R}$ and ${}^\rho\mathbb{R}$, repsectively. We shall write always $\| \ \|$ for the norms of $\mathbb{R}^d$, ${}^*\mathbb{R}^d$ and ${}^\rho\mathbb{R}^d$, leaving to the reader to decide from the context which of these norms we have in mind. Notice that $\| \ \|$ takes non-negative values in $\mathbb{R}$, ${}^*\mathbb{R}$ and ${}^\rho\mathbb{R}$, respectively. In the case $d = 1$ this norm reduces to the absolute value in $\mathbb{R}$, ${}^*\mathbb{R}$ and ${}^\rho\mathbb{R}$, respectively. It is clear that $\mathbb{R}^d$ is a vector subspace of both ${}^*\mathbb{R}^d$ and ${}^\rho\mathbb{R}^d$ over $\mathbb{R}$, in symbols, $\mathbb{R}^d \subset {}^*\mathbb{R}^d$ and $\mathbb{R}^d \subset {}^\rho\mathbb{R}^d$. We call the elements in ${}^\rho\mathbb{R}^d$ *asymptotic points* or *asymptotic vectors*. Unless it is said otherwise, we shall always topologize ${}^\rho\mathbb{R}^d$ by the *product-interval topology*, i.e. by the product topology in ${}^\rho\mathbb{R}^d$, generated by the order topology in ${}^\rho\mathbb{R}$. The sequence of balls:

$$B_n = \{x \in {}^\rho\mathbb{R}^d : \|x\| < s^n\}, \quad n \in \mathbb{N}, \tag{1.5}$$

forms a base for the neighborhoods of the zero in ${}^\rho\mathbb{R}^d$.

Let $x \in {}^\rho\mathbb{R}^d$. Then x is called *infinitesimal, finite or infinitely large* if its ρ-norm $\|x\|$ is infinitesimal, finite or infinitely large in ${}^\rho\mathbb{R}$, respectively. We denote by $\mathcal{F}({}^*\mathbb{R}^d)$ and $\mathcal{F}({}^\rho\mathbb{R}^d)$ the sets of the finite elements of ${}^*\mathbb{R}^d$ and ${}^\rho\mathbb{R}^d$, respectively, and by $\mathcal{I}({}^*\mathbb{R}^d)$ and $\mathcal{I}({}^\rho\mathbb{R}^d)$ we denote the sets of the infinitesimals points in ${}^*\mathbb{R}^d$ and ${}^\rho\mathbb{R}^d$, respectively. If Ω is an open set of $\mathbb{R}^d$, then $\tilde{\Omega} = \Omega + \mathcal{I}({}^*\mathbb{R}^d)$ denotes the set of the *nearstandard points of* ${}^*\Omega$ in ${}^*\mathbb{R}^d$ and ${}^\rho\tilde{\Omega} = \Omega + \mathcal{I}({}^\rho\mathbb{R}^d)$ denotes the set of the *nearstandard points of* ${}^*\Omega$ in ${}^\rho\mathbb{R}^d$, in symbols,

$$\tilde{\Omega} = \Omega + \mathcal{I}({}^*\mathbb{R}^d) = \{x + \xi \ : \ x \in \Omega, \ \xi \in \mathcal{I}({}^*\mathbb{R}^d)\}, \tag{1.6}$$

$${}^\rho\tilde{\Omega} = \Omega + \mathcal{I}({}^*\mathbb{R}^d) = \{x + h \ : \ x \in \Omega, \ h \in \mathcal{I}({}^*\mathbb{R}^d)\}, \tag{1.7}$$

where the sum in $x+\xi$ is in ${}^*\mathbb{R}^d$, while the sum in $x+h$ is in ${}^\rho\mathbb{R}^d$. We observe that $\Omega \subset \tilde{\Omega} \subseteq {}^*\Omega$ and $\Omega \subset {}^\rho\tilde{\Omega} \subseteq {}^\rho\Omega$. In the particular case $\Omega = \mathbb{R}^d$ we have

$$\mathbb{R}^d + \mathcal{I}({}^*\mathbb{R}^d) = \mathcal{F}({}^*\mathbb{R}^d) \text{ and } \mathbb{R}^d + \mathcal{I}({}^\rho\mathbb{R}^d) = \mathcal{F}({}^\rho\mathbb{R}^d). \tag{1.8}$$

We denote by $({}^\rho\mathbb{C})^{{}^\rho\tilde{\Omega}}$ or $({}^\rho\mathbb{C})^{\Omega+\mathcal{I}({}^\rho\mathbb{R}^d)}$ the ring (under the pointwise operations) of the functions from ${}^\rho\tilde{\Omega}$ into ${}^\rho\mathbb{C}$.

The purpose of the next several lines is to connect the sets $\tilde{\Omega}$ and ${}^\rho\tilde{\Omega}$. First, we observe that the vector space ${}^\rho\mathbb{R}^d$ can, alternatively, be constructed as the factor space ${}^\rho\mathbb{R}^d = \mathcal{M}_\rho({}^*\mathbb{R}^d)/\mathcal{N}_\rho({}^*\mathbb{R}^d)$, where

$$\mathcal{M}_\rho({}^*\mathbb{R}^d) = \{\xi \in^* \mathbb{R}^d : \|\xi\| < \rho^{-n} \quad \text{for some} \quad n \in \mathbb{N}\}, \tag{1.9}$$

$$\mathcal{N}_\rho({}^*\mathbb{R}^d) = \{x \in^* \mathbb{R}^d : \|x\| < \rho^{-n} \quad \text{for all} \quad n \in \mathbb{N}\}, \tag{1.10}$$

are the sets of the ρ-moderate and ρ-null points in ${}^*\mathbb{R}^d$, respectively. We denote by $q^d : \mathcal{M}_\rho({}^*\mathbb{R}^d) \to {}^\rho\mathbb{R}^d$ the corresponding quotient mapping. We let $q^1 = q$ in the case $d = 1$. The imbedding $\mathbb{R}^d \subset {}^\rho\mathbb{R}^d$ is reproduced by by $x \to q^d(x)$. If $\mathcal{A} \subseteq {}^*\mathbb{R}^d$, we let

$$\mathcal{M}_\rho(\mathcal{A}) = \{\xi \in \mathcal{A} : \|\xi\| < \rho^{-n} \quad \text{for some} \quad n \in \mathbb{N}\}, \tag{1.11}$$

and ${}^\rho\mathcal{A} = q[\mathcal{M}_\rho(\mathcal{A})]$. The set ${}^\rho\mathcal{A} \subseteq {}^\rho\mathbb{R}^d$ is called the ρ-extension of $\mathcal{A}$ in ${}^\rho\mathbb{R}^d$. In the particular case $\mathcal{A} = {}^*A$, where $A \subseteq \mathbb{R}^d$, we shall write simply ${}^\rho A$ instead of the more precise ${}^\rho({}^*A)$. We observe that ${}^\rho\tilde{\Omega}$ is the ρ-extension of $\tilde{\Omega}$ and also we have:

$$\xi \in \tilde{\Omega} \Leftrightarrow q^d(\xi) \in {}^\rho\tilde{\Omega}. \tag{1.12}$$

For more details about the vector spaces over the field ${}^\rho\mathbb{R}$ we refer to W.A.J. Luxemburg [7].

Part 3. *Asymptotic Functions*: The algebra of *asymptotic functions* ${}^\rho\mathcal{E}(\Omega)$ on Ω is the factor space ${}^\rho\mathcal{E}(\Omega) = \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))/\mathcal{N}\rho({}^*\mathcal{E}(\Omega))$, where

$$\mathcal{M}_\rho({}^*\mathcal{E}(\Omega)) = \{f \in {}^*\mathcal{E}(\Omega) : (\forall\alpha \in \mathbb{N}_0^d)(\forall\xi \in \tilde{\Omega})(\partial^\alpha f(\xi) \in \mathcal{M}_\rho({}^*\mathbb{C}))\}, \tag{1.13}$$

$$\mathcal{N}_\rho({}^*\mathcal{E}(\Omega)) = \{f \in {}^*\mathcal{E}(\Omega) : (\forall\alpha \in \mathbb{N}_0^d)(\forall\xi \in \tilde{\Omega})(\partial^\alpha f(\xi) \in \mathcal{N}_\rho({}^*\mathbb{C}))\}. \tag{1.14}$$

The functions in $\mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$ are called ρ-moderate (or, simply, *moderate)* and those in $\mathcal{N}_\rho({}^*\mathcal{E}(\Omega))$ are called *ρ-null functions* (or, simply, *null functions*). The *canonical (differential ring) imbedding* $\mathcal{E}(\Omega) \subset {}^\rho\mathcal{E}(\Omega)$ is defined by $f \to \sigma(f)$, where $\sigma(f) = Q_\Omega({}^*f)$ and $Q_\Omega : \mathcal{M}_\rho({}^*\mathcal{E}(\Omega)) \to {}^\rho\mathcal{E}(\Omega)$ is the quotient mapping. For the construction of an imbedding of the Schwartz distributions: $\mathcal{E}(\Omega) \subset \mathcal{D}'(\Omega) \subset {}^\rho\mathcal{E}(\Omega)$, discussed in the Introduction of this paper, we refer to (M. Oberguggenberger and T. Todorov [10]).

2 Asymptotic Functions as Pointwise Functions

We define an evaluation of the asymptotic functions in ${}^\rho\mathcal{E}(\Omega)$ and show that this evaluation coincides with the usual evaluation in the particular case of classical smooth functions. Moreover, we construct a ring imbedding of ${}^\rho\mathcal{E}(\Omega)$ into the ring of pointwise functions $({}^\rho\mathbb{C})^{{}^\rho\tilde{\Omega}}$, thus, showing that the asymptotic functions can be characterized as pointwise functions. In particular, the Schwartz distributions also can be characterized by their values. In what follows we shall use the notations in Section 1.

Recall that every asymptotic function is, by definition, an equivalence class of nonstandard internal functions in ${}^*\mathcal{E}(\Omega)$ (Section 1, Part 3). On the other hand, the functions in ${}^*\mathcal{E}(\Omega)$ are pointwise functions from ${}^*\Omega$ into ${}^*\mathbb{C}$. We should notice, however, that the asymptotic functions $Q_\Omega(f)$ can not inherit literally the values of its representatives f because the factorization in $Q_\Omega(f)$ destroys this evaluation.

(2.1) Definition *(Values)*: Let $F = Q(f)$ be an asymptotic function in ${}^\rho\mathcal{E}(\Omega)$ with a representative $f \in \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$. Let $x = q(\xi)$ be an asymptotic point in ${}^\rho\tilde{\Omega}$ with a representative $\xi \in \tilde{\Omega}$. The asymptotic number $q(f(\xi)) \in {}^\rho\mathbb{C}$ is called the *value* of F at x, in symbols,

$$F(x) = q(f(\xi)) \quad \textit{or} \quad Q(f)(q(\xi)) = q(f(\xi)).$$

We should mention that the evaluation of f at ξ is in the framework of ${}^*\mathbb{C}$.

(2.2) Lemma*(Correctness)*: Let $\alpha,\, \beta \in \tilde{\Omega},\ \|\alpha - \beta\| \in \mathcal{N}_\rho({}^*\mathbb{R})$ and $f,\, g \in \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$, $f - g \in \mathcal{N}_\rho({}^*\mathcal{E}(\Omega))$. Then $f(\alpha) - g(\beta) \in \mathcal{N}_\rho({}^*\mathbb{C})$ (or, equivalently, $|f(\alpha) - g(\beta)| \in \mathcal{N}_\rho({}^*\mathbb{R})$).

Proof: By the Mean Value Theorem, applied by Transfer Principle (T. Todorov [15], p. 686), there exists $\gamma \in {}^*\mathbb{R}^d$ between α and β (in the sense that $\gamma = \alpha + \varepsilon(\beta - \alpha)$ for some $\varepsilon \in {}^*\mathbb{R}, 0 \leq \epsilon \leq 1$) such that

$$|f(\alpha) - f(\beta)| = |\nabla f(\gamma) \cdot (\alpha - \beta)| \leq \|\nabla f(\gamma)\| \, \|\alpha - \beta\| \, .$$

The point γ is in $\tilde{\Omega}$, since $\alpha \approx \beta$, by assumption, thus, $\|\nabla f(\gamma)\|$ is a moderate number in $\mathcal{M}_\rho({}^*\mathbb{R})$, by the definition of $\mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$. It follows that the right hand side of the above inequality is in $\mathcal{N}_\rho({}^*\mathbb{R})$, since $\mathcal{N}_\rho({}^*\mathbb{R})$ is an ideal in $\mathcal{M}_\rho({}^*\mathbb{R})$. On the other hand, we have

$$\|f(\alpha) - g(\beta)\| = \|f(\alpha) - f(\beta) + f(\beta) - g(\beta)\| \leq \|f(\alpha) - f(\beta)\| +$$

$$+ \|f(\beta) - g(\beta)\| \leq \|\nabla f(\gamma)\| \, \|\alpha - \beta\| + \|f(\beta) - g(\beta)\| \in \mathcal{N}_\rho({}^*\mathbb{R}),$$

as required, since $\|f(\beta) - g(\beta)\| \in \mathcal{N}_\rho({}^*\mathbb{R})$, by assumption. □

Recall that the algebra of standard smooth functions $\mathcal{E}(\Omega)$ is imbedded in ${}^\rho\mathcal{E}(\Omega)$ in a canonical way, in symbols, $\mathcal{E}(\Omega) \subset {}^\rho\mathcal{E}(\Omega)$, by the mapping $f \to \sigma(f)$, where $\sigma(f) = Q({}^*f)$. Recall as well that $\mathbb{C} \subset {}^\rho\mathbb{C}$ by the mapping $c \to q(c)$. In what follows we shall identify c with its image $q(c)$, writing simply $c = q(c)$.

The next result shows that the concept of *value* for the asymptotic functions, introduced above, is a generalization of the usual evaluation of the classical functions.

(2.3) Lemma *(Standard Functions at Standard Points)*: If $f \in \mathcal{E}(\Omega)$ and $x \in \Omega$, then $\sigma(f)(x) = f(x)$.

Proof: $\sigma(f)(x) = Q({}^*f)(q(x)) = q({}^*f(x)) = q(f(x)) = f(x)$ since ${}^*f$ is an extension of f and $f(x)$ is a standard (complex) number. □

The next two lemmas are in the framework of ${}^*\mathbb{R}^d$ and ${}^*\mathbb{C}$.

(2.4) Lemma: Let $f \in \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$ and $\xi \in \tilde{\Omega}$ and $\chi \in \mathcal{N}_\rho({}^*\mathbb{R}^d)$. Then:

(i) $\|f(\xi + \chi) - f(\xi)\| \in \mathcal{N}_\rho({}^*\mathbb{C})$.

(ii) $\dfrac{\|f(\xi + \chi) - f(\xi) - \nabla f(\xi) \cdot \chi\|}{\|\chi\|} \in \mathcal{N}_\rho({}^*\mathbb{C}), \quad \chi \neq 0.$

Proof: (i) By the Mean Value Theorem, applied by Transfer Principle (T. Todorov [15], p. 686) there exists $\varepsilon \in {}^*\mathbb{R}$, $0 \leq \varepsilon \leq 1$, such that

$$\|f(\xi + \chi) - f(\xi)\| = \|\nabla f(\xi + \varepsilon\chi) \cdot \chi\|$$

It follows $\|f(\xi+\chi)-f(\xi)\| \leq \|\nabla f(\xi+\varepsilon\chi)\| \; \|\chi\|$. Notice that $\xi+\varepsilon\chi$ is also in $\tilde{\Omega}$ (since $\varepsilon\chi$ is an infinitesimal point) which implies $\|\nabla f(\xi+\varepsilon\chi)\| \in \mathcal{M}_\rho({}^*\mathbb{R})$ since $f \in \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$, by assumption. Now, the result follows since $\mathcal{N}_\rho({}^*\mathbb{R})$ is an ideal in $\mathcal{M}_\rho({}^*\mathbb{R})$.

(ii) By the Taylor Theorem, applied by Transfer Principle, there exists $\varepsilon \in {}^*\mathbb{R}$, $0 \leq \varepsilon \leq 1$, such that

$$\frac{\|f(\xi+\chi)-f(\xi)-\nabla f(\xi)\cdot\chi\|}{\|\chi\|} = \frac{1}{2}\left\|\sum_{|\alpha|=2} \partial^\alpha f(\xi+\varepsilon\chi)\frac{\chi^\alpha}{\|\chi\|}\right\|$$

As before, $\|\partial^\alpha f(\xi+\varepsilon\chi)\| \in \mathcal{M}_\rho({}^*\mathbb{R})$ and also we have $\chi^\alpha/\|\chi\| \in \mathcal{N}_\rho({}^*\mathbb{R})$ due to $|\alpha| = 2$. Thus, it follows that the right hand side of the above equality is in $\mathcal{N}_\rho({}^*\mathbb{R})$, since $\mathcal{N}_\rho({}^*\mathbb{R})$ is an ideal in $\mathcal{M}_\rho({}^*\mathbb{R})$. $\square$

In what follows $K \subset\subset \Omega$ means that K is a compact subset of Ω.

(2.5) Lemma: Let $f \in \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$. Then:

(i) $(\forall K \subset\subset \Omega)(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(\forall \xi \in {}^*K)(\forall \chi \in {}^*\mathbb{R})$
$[(\|\chi\| < \rho^m) \Rightarrow (\|f(\xi+\chi)-f(\xi)\| < \rho^n)]$.

(ii) $(\forall K \subset\subset \Omega)(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(\forall \alpha \in {}^*K)(\forall \chi \in {}^*\mathbb{R})$
$$\left[(0 < \|\chi\| < \rho^m) \Rightarrow \quad \frac{\|f(\xi+\chi)-f(\xi)-\nabla f(\xi)\cdot\chi\|}{\|\chi\|} \; < \; \rho^n\right].$$

Proof: (i) Suppose (for contradiction) that (i) fails, i.e.

$$(\exists K \subset\subset \Omega)(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(S_m \neq \emptyset),$$

where

$$S_m = \left\{(\xi,\chi) \in {}^*K \times {}^*\mathbb{R}^d : (\|\chi\| < \rho^m) \quad and \quad (\|f(\xi+\chi)-f(\xi)\| \geq \rho^n)\right\}.$$

It is clear that $S_m \supset S_{m+1} \supset ...$, hence, by the Saturation Principle (T. Todorov [15], p. 687), there exists a pair $(\xi_0, \chi_0) \in S_m$ for all $m \in \mathbb{N}$. The latter means $\|\chi_0\| \in \mathcal{N}_\rho({}^*\mathbb{R})$ and $\|f(\xi_0+\chi_0)-f(\chi_0)\| \notin \mathcal{N}_\rho({}^*\mathbb{R})$, contradicting the assumption $f \in \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$ in view of Lemma 2.4.

(ii) As above, suppose (for contradiction) that (ii) fails, i.e. $(\exists K \subset\subset \Omega)(\exists n \in$

$\mathbb{N})(\forall m \in \mathbb{N})(T_m \neq \emptyset)$, where

$$T_m = \Big\{(\xi, \chi) \in {}^*K \times {}^*\mathbb{R}^d : \quad 0 < \|\chi\| < \rho^m \quad \text{and} \quad \frac{\|f(\xi+\chi) - f(\xi) - \nabla f(\xi)\cdot\chi\|}{\|\chi\|} \geq \rho^n\Big\}.$$

As before, we observe that $T_m \subset T_{m+1} \dots$, hence, by the Saturation Principle, there exists a pair $(\xi_0, \chi_0) \in T_m$ for all $m \in \mathbb{N}$. The latter means $\|\chi_0\| \in \mathcal{N}_\rho({}^*\mathbb{R})$ and

$$\frac{\|f(\xi_0+\chi_0) - f(\xi_0) - \nabla f(\xi_0)\cdot\chi_0\|}{\|\chi_0\|} \notin \mathcal{N}_\rho({}^*\mathbb{R}),$$

contradicting the assumption $f \in \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$ in view of Lemma 2.4. □

(2.6) Theorem: Let $F \in {}^\rho\mathcal{E}(\Omega)$, $\partial^\alpha F$ be its partial derivatives and ∇F be the corresponding gradient of F in ${}^\rho\mathcal{E}(\Omega)$. Let $x \in {}^\rho\tilde{\Omega}$ be a finite asymptotic number and $F(x)$, $(\partial^\alpha F)(x)$ and $(\nabla F)(x)$ be the values of F, $\partial^\alpha F$ and ∇F at the point x, respectively. Then:

(i) $\lim\limits_{{}^\rho\mathbb{R}^d \ni h \to 0} F(x+h) = F(x)$.

(ii) $\lim\limits_{{}^\rho\mathbb{R}^d \ni h \to 0} \dfrac{\|F(x+h) - F(x) - (\nabla F)(x)\cdot h\|}{\|h\|} = 0$

where both limits are in the product-interval topology of ${}^\rho\mathbb{R}^d$ (1.5).

Proof: We have $F = Q(f), x = q(\xi)$ and $h = q(\chi)$ for some $f \in \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$ some $\xi \in \tilde{\Omega}$ and some $\chi \in \mathcal{M}_\rho({}^*\mathbb{R}^d)$. Now, both (i) and (ii) follow immediately from the above lemma after taking quotient mappings Q_Ω and q^d, taking into account that the sequence of balls $\{B_n\}$ (1.5) forms a base for the neighborhoods of the zero in the product-interval topology in ${}^\rho\mathbb{R}^d$. □

(2.7) Corollary (*The Case* $\Omega = \mathbb{R}$): Let $F \in {}^\rho\mathcal{E}(\mathbb{R})$ and F' be its derivative in the algebra ${}^\rho\mathcal{E}(\mathbb{R})$. Let $x \in \mathcal{F}({}^\rho\mathbb{R})$ be a finite asymptotic number and $F(x)$ and $F'(x)$ be the values of F and F' at the point x, respectively. Then:

(i) $\lim\limits_{{}^\rho\mathbb{R}^d \ni h \to 0} F(x+h) = F(x)$.

(ii) $\lim\limits_{{}^\rho\mathbb{R}^d \ni h \to 0} \dfrac{F(x+h) - F(x)}{h} = F'(x)$

where both limits are in the interval topology of ${}^\rho\mathbb{R}$ (1.4) (or, equivalently, in the valuation-metric topology of ${}^\rho\mathbb{R}$).

Proof: The result follows as a particular case from the above theorem for $d = 1$ and $\Omega = R$, taking into account that in this case we have $\widetilde{\mathbb{R}} = \mathbb{R} + \mathcal{I}(^*\mathbb{R}) = \mathcal{F}(^*\mathbb{R})$ and $^\rho\widetilde{\mathbb{R}} = \mathbb{R} + \mathcal{I}(^\rho\mathbb{R}) = \mathcal{F}(^\rho\mathbb{R})$. $\square$

(2.8) Theorem *(The Imbedding)*: Define $V : {}^\rho\mathcal{E}(\mathbb{R}) \to ({}^\rho\mathbb{C})^{{}^\rho\tilde{\Omega}}$, by $F \to V(F)$, where $V(F)(x) = F(x)$, $x \in {}^\rho\tilde{\Omega}$. Then V is a ring imbedding in the sense that it is injective and it preserves the addition and multiplication.

Proof: Suppose that $V(F) = 0$ in $({}^\rho\mathbb{C})^{{}^\rho\tilde{\Omega}}$, i.e. $F(x) = 0$ in ${}^\rho\mathbb{C}$ for all $x \in {}^\rho\tilde{\Omega}$. We have to show that $F = 0$ in the algebra ${}^\rho\mathcal{E}(\Omega)$, i.e. to show that $F = Q_\Omega(f)$ for some $f \in {}^*\mathcal{E}(\mathbb{R})$ such that $\partial^\alpha f(\xi) \in \mathcal{N}_\rho({}^*\mathbb{C})$ for all $\xi \in \tilde{\Omega}$ and all $\alpha \in \mathbb{N}_0^d$ (1.14)). By our assumption we have $F(x+h) - F(x) = 0$ for all $x \in {}^\rho\tilde{\Omega}$ and all $h \in \mathcal{I}({}^\rho\mathbb{R}^d)$ (since $x + h \in {}^\rho\tilde{\Omega}$ which implies $F(x+h) = 0$ along with $F(x) = 0$). It follows that $\|\nabla F(x)\| = 0$ for all $x \in^\rho \tilde{\Omega}$, by the second part of Theorem 2.6. In other words, $\partial^\alpha F(x) = 0$ for all $x \in {}^\rho\tilde{\Omega}$ and all $\alpha \in \mathbb{N}_0^d$. Next, observe that each partial derivative $\partial^\alpha F$ with $|\alpha| = 1$, also belongs to ${}^\rho\mathcal{E}(\Omega)$, so, we can repeat the above arguments by induction in $|\alpha|$; the result is $(\partial^\alpha F)(x) = 0$ in ${}^\rho\mathbb{C}$ for all $x \in {}^\rho\tilde{\Omega}$ and all $\alpha \in \mathbb{N}_0^d$. On the other hand, we have $F = Q_\Omega(f)$ for some $f \in \mathcal{M}_\rho({}^*\mathcal{E}(\mathbb{R}))$ and $\partial^\alpha F = Q_\Omega(\partial^\alpha f)$. Also, $x \in {}^\rho\tilde{\Omega}$ means $x = q^d(\xi)$ for some $\xi \in \tilde{\Omega}$ (1.12). Hence, we conclude that $q(\partial^\alpha f(\xi)) = 0$ for all $\xi \in \tilde{\Omega}$ and all $\alpha \in \mathbb{N}_0^d$. The latter means $\partial^\alpha f(\xi) \in \mathcal{N}_\rho({}^*\mathbb{C})$ for all $\xi \in \tilde{\Omega}$ and all $\alpha \in \mathbb{N}_0^d$, as required. To show the preservation of the ring operations, let us take another asymptotic function $G = Q(g)$ with a representative $g \in \mathcal{M}_\rho({}^*\mathcal{E}(\mathbb{R}))$. For the addition we have

$$(F+G)(x) = Q(f+g)(q(\xi)) = q(f(\xi) + g(\xi)) = q(f(\xi)) + q(g(\xi)) =$$

$$= Q(f)(q(\xi)) + Q(g)(q(\xi)) = F(x) + G(x),$$

and similar for the multiplication. $\square$

(2.9) Corollary (C^∞*-Functions)*: Let $F \in {}^\rho\mathcal{E}(\Omega)$ and $V(F)$ be its image in $({}^\rho\mathbb{C})^{{}^\rho\tilde{\Omega}}$. Then $V(F)$ is a pointwise C^∞-function from ${}^\rho\tilde{\Omega}$ to ${}^\rho\mathbb{C}$ in the sense that for any $x \in {}^\rho\tilde{\Omega}$ and any $\alpha \in \mathbb{N}_0^d$ the partial derivative $\partial^\alpha V(F)$ exists and it is continuous at x in the product-order topology in ${}^\rho\mathbb{R}^d$ (1.5).

Proof: The result follows directly from Theorem 2.6, by induction in $|\alpha|$. $\square$

3 Fundamental Theorem of Calculus in ${}^\rho\mathcal{E}(\Omega)$

We prove the Fundamental Theorem of Integration Calculus in ${}^\rho\mathcal{E}(\Omega)$ and show that the set of the scalars of ${}^\rho\mathcal{E}(\Omega)$ coincides exactly with A. Robinson's field ${}^\rho\mathbb{C}$ and the set of the real scalars of ${}^\rho\mathcal{E}(\Omega)$ coincides exactly with A. Robinson's field ${}^\rho\mathbb{R}$.

(3.1) Definition (*Scalars in* ${}^\rho\mathcal{E}(\Omega)$): We call $F \in {}^\rho\mathcal{E}(\Omega)$ a scalar in ${}^\rho\mathcal{E}(\Omega)$ if $\partial^\alpha F = 0$ in ${}^\rho\mathcal{E}(\Omega)$ for all $\alpha \in \mathbb{N}_0^d$, $|a| = 1$, in short, if $\nabla F = 0$ in ${}^\rho\mathcal{E}(\Omega)$.

(3.2) Theorem (*Fundamental Theorem in* ${}^\rho\mathcal{E}(\Omega)$): Let Ω be an open subset of $\mathbb{R}^d$ which in addition is arcwise connected. Let $F \in^\rho \mathcal{E}(\Omega)$ be an asymptotic function and ∇F be its gradient in ${}^\rho\mathcal{E}(\Omega)$. Then the following are equivalent:

(i) F is a scalar in ${}^\rho\mathcal{E}(\Omega)$, i.e. $\nabla F = 0$ in ${}^\rho\mathcal{E}(\Omega)$.

(ii) $(\nabla F)(x) = 0$ in ${}^\rho\mathbb{C}^d$ for all $x \in {}^\rho\tilde{\Omega}$ (pointwisely) in the sense that $(\partial^\alpha F)(x) = 0$ in ${}^\rho\mathbb{C}$ for all $x \in {}^\rho\mathcal{E}(\Omega)$ and all $\alpha \in \mathbb{N}_0^d$, $|\alpha| = 1$.

(iii) F is a constant asymptotic function in the sense that $F(x) = C$ in ${}^\rho\mathbb{C}$ for some $C \in {}^\rho\mathbb{C}$ and all $x \in {}^\rho\tilde{\Omega}$.

(iv) $F = Q_\Omega(c)$ in ${}^\rho\mathcal{E}(\Omega)$ for some $c \in \mathcal{M}_\rho({}^*\mathbb{C})$, more precisely, $F = Q_\Omega(f_c)$, defined by $f_c(\xi) = c$ for all $\xi \in {}^*\Omega$.

The connection between (iii) and (iv) is given by $C = q(c)$, where $q : \mathcal{M}_\rho({}^*\mathbb{C}) \to {}^\rho\mathbb{C}$ is the quotient mapping (Section 1, Part 1).

Proof: (i) $\Leftrightarrow$ (ii) follows immediately from Theorem 2.8.

(ii) $\Rightarrow$ (iii): Observe that $V(\|\nabla F\|) = 0$ in $({}^\rho\mathbb{C})^{{}^\rho\tilde{\Omega}}$ implies $\|\nabla F\| = 0$ in ${}^\rho\mathcal{E}(\Omega)$, by Theorem 2.8. Let $x_1, x_2 \in {}^\rho\tilde{\Omega}$. We have to show that $F(x_1) = F(x_2)$ in ${}^\rho\mathbb{C}$. We need the representatives: we have $F = Q_\Omega(f)$ for some $f \in \mathcal{M}_\rho((\mathcal{E}(\Omega))$ and $x_i = q^d(\xi_i)$, for some $\xi_i \in \tilde{\Omega}$, $i = 1, 2$. Since Ω is arcwise connected, it follows that ${}^*\Omega$ is arcwise connected, by Transfer Principle (T. Todorov [15], p. 686). Hence, ξ_1 and ξ_2 can be connected with a $*$-continuous curve $L \subset {}^\rho\tilde{\Omega}$. It follows

$$f(\xi_2) - f(\xi_1) = \int_L \nabla f(\xi) \cdot d\xi,$$

by Transfer Principle. Continuing the arguments, $\|\nabla F\| = 0$ in ${}^\rho\mathcal{E}(\Omega)$ implies $\|\nabla f\| \in \mathcal{N}_\rho({}^*\mathcal{E}(\Omega))$. Thus,

$$|f(\xi_2) - f(\xi_1)| \leq \|\nabla f(\xi_0)\| \|\xi_2 - \xi_1\| \in \mathcal{N}_\rho({}^*\mathbb{R})$$

where ξ_0 is some point on L. It follows $f(\xi_2) - f(\xi_1) \in \mathcal{N}_\rho({}^*\mathbb{C})$. Hence, $F(x_2) - F(x_1) = q(f(\xi_2) - f(\xi_1)) = 0$, as required.

(ii) $\Leftarrow$ (iii): We have $F(x + h) - F(x) = 0$ for all $x \in {}^\rho\tilde{\Omega}$ and all $h \in \mathcal{I}({}^\rho\mathbb{R}^d)$, by assumption (since $x + h$ is also in ${}^\rho\tilde{\Omega}$). It follows $\nabla F(x) = 0$ for all $x \in {}^\rho\tilde{\Omega}$, by the (ii) - part of Theorem 2.6.

(iii) $\Rightarrow$ (iv): Suppose that $F = Q_\Omega(f)$ for some $f \in \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$ and $C = q(c)$ for some $c \in \mathcal{M}_\rho({}^*\mathbb{C})$. Now, $F(x) = C$ for all $x \in {}^\rho\tilde{\Omega}$ implies $q(f(\xi) - c) = 0$ or, equivalently, $f(\xi) - c \in \mathcal{N}_\rho({}^*\mathbb{C})$, for all $\xi \in \tilde{\Omega}$ (1.14). Hence, $F - Q_\Omega(f_c) = Q_\Omega(f - f_c) = Q_\Omega(f - c) = 0$, as required.

(iii) $\Leftarrow$ (iv): We have $F(x) = q(f_c(\xi)) = q(c) = C$, where $x = q(\xi)$. $\square$

(3.3) Corollary:

(i) The set of the scalars in ${}^\rho\mathcal{E}(\Omega)$:

$$\{F \in^\rho \mathcal{E}(\Omega) : \nabla F = 0 \quad in \quad {}^\rho\mathcal{E}(\Omega)\}$$

is an algebraically closed non-archimedean field which is isomorphic to the field of A. Robinson's complex asymptotic numbers ${}^\rho\mathbb{C}$ (Section 1, Part 1) under the imbedding $\mathcal{C} : F \to F(x)$, where x is an arbitrary point in ${}^\rho\tilde{\Omega}$. Consequently, ${}^\rho\mathbb{C}$ is imbedded in ${}^\rho\mathcal{E}(\Omega)$, in symbols, ${}^\rho\mathbb{C} \subset {}^\rho\mathcal{E}(\Omega)$, by $\mathcal{C} : C \to Q_\Omega(f_c)$, where $f_c \in {}^*\mathcal{E}(\Omega)$ is defined by $f_c(\xi) = c$ for all $\xi \in {}^*\Omega$, where $C = q(c)$. The imbedding $\mathcal{C}^{-1}$ preserves the addition and multiplication.

(ii) The set of the real scalars in ${}^\rho\mathcal{E}(\Omega)$:

$$\{\pm|F| : F \in {}^\rho\mathcal{E}(\Omega) \quad \text{such that} \quad \nabla F = 0 \quad \text{in} \quad {}^\rho\mathcal{E}(\Omega)\}$$

is a totally ordered real closed Cantor complete non-archimedean field which is isomorphic to the field of A. Robinson's real asymptotic numbers ${}^\rho\mathbb{R}$ (Section 1, Part 1) under the imbedding $\mathcal{C} : \pm|F| \to \pm|F(x)|$, where x is an arbitrary point in ${}^\rho\tilde{\Omega}$.

Proof: (i) For every scalar F in ${}^{\rho}\mathcal{E}(\Omega)$ there exists $C \in {}^{\rho}\mathbb{C}$ such that $F(x) = C$ for all $x \in {}^{\rho}\tilde{\Omega}$, by Theorem 3.2. So, the mapping $F \to F(x)$ from the constant functions in ${}^{\rho}\mathcal{E}(\Omega)$ into ${}^{\rho}\mathbb{C}$ is well defined. The preservation of the addition and multiplication follows from Theorem 2.8. Conversely, if $C = q(c)$ is in ${}^{\rho}\mathbb{C}$, then the function $F = Q_{\Omega}(f_c)$ is the preimage of C.

(ii) follows immediately from (i) taking into account that ${}^{\rho}\mathbb{R} = \{\pm|a| : \alpha \in {}^{\rho}\mathbb{C}\}$. $\square$

Acknowledgment: The author thanks Michael Oberguggenberger for the discussion of the preliminary version of this paper.

References

[1] J. F. Colombeau, New Generalized Functions and Multiplication of Distributions, North-Holland Math. Studies 84, North-Holland, Amsterdam 1984.

[2] J. F. Colombeau, Elementary Introduction to New Generalized Functions, North-Holland Math. Studies 113, North-Holland, Amsterdam 1985.

[3] R. F. Hoskins, J. Sousa Pinto, Nonstandard Treatments of New Generalised Functions, In: R.S. Pathak (Ed.), Generalized Functions and Their Applications, Plenum Press, New York 1993, 95-108.

[4] Li Bang-He, Non-standard analysis and multiplication of distributions, Sci. Sinica **21**(1978), 561-585.

[5] T. Lindstrøm, An Invitation to Nonstandard Analysis. In: N. Cutland (Ed.), Nonstandard Analysis and its Applications, London Mathematical Society Student Texts 10, Cambridge University Press, Cambridge 1988, 1- 105.

[6] A. H. Lightstone, A. Robinson, Nonarchimedean Fields and Asymptotic Expansions, North-Holland Math. Library 13, North-Holland, Amsterdam; American Elsevier, New York 1975.

[7] W. A. J. Luxemburg, On a class of valuation fields introduced by A. Robinson, Israel J. Math. **25**(1976), 189 - 201.

[8] M. Oberguggenberger, Products of distributions: nonstandard methods, Z. Anal. Anwendungen **7**(1988), 347 - 365. Corrections: ibid. **10**(1991), 263-264.

[9] M. Oberguggenberger, Contributions of Nonstandard Analysis to Partial Differential Equations. In: N. J. Cutland, V. Neves, F. Oliveira, J. Sousa-Pinto (Eds.), Developments in Nonstandard Mathematics, Longman, Harlow 1995, 130-150.

[10] M. Oberguggenberger, T. Todorov, An embedding of Schwartz distributions in the algebra of asymptotic functions, Int. J. Math. & Math. Sci. **21**(1998), 417-428.

[11] M. Kunzinger and M. Oberguggenberger, *Characterization of Colombeau generalized functions by their pointvalues*, Math. Nachr., 203, 147-157, 1999.

[12] V. Pestov, On a valuation field invented by A. Robinson and certain structures connected with it. Israel J. Math. **74**(1991), 65-79.

[13] A. Robinson, Function theory on some nonarchimedean fields, Amer. Math. Monthly **80**(6)(1973), Part II: Papers in the Foundations of Mathematics, 87 - 109.

[14] T. Todorov, Colombeau's New Generalized Functions and Non-Standard Analysis, In: B. Stanković, E. Pap, S. Pilipović, V. S. Vladimirov (Eds.), Generalised Functions, Convergence Structures and Their Applications, Plenum Press, New York 1988, 327 - 339.

[15] T. Todorov, An existence result for linear partial differential equations with $\mathcal{C}^\infty$-coefficients in an algebra of generalized functions, Trans. Am. Math. Soc. **348**(1996), 673 - 689.

[16] V. S. Vladimirov, Generalized Functions in Mathematical Physics, Mir-Publishers, Moscow, 1979.

T. D. Todorov
California Polytechnic State University, Department of Mathematics, San Luis Obispo, Califirnia 93407, USA, E-mail: ttodorov@polymail.calpoly.edu